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A Plea for Natural Philosophy and Other Essays

Enhanced If-thenism

Perhaps the most fundamental question in the philosophy of mathematics is simply this: what are we doing when we engage in mathematical thinking? For centuries of recorded history, the developing answers to this question were straightforward. Mathematics began with the ancients in purely practical problems -- we were measuring lengths and volumes, exchanging money, farming and building, studying the stars, constructing calendars -- and gradually expanded into a complex study of the underlying structure of the physical world. This way of understanding the subject continued through the profound developments of the 17^{th} century -- Descartes, Galileo (with his ‘great book of Nature’, written in the language of mathematics), Newton, Leibniz -- and into the 18^{th} -- with Euler the first among many. But this near-identification of mathematics with natural science was gradually overthrown in the course of the 19^{th} century, with increasing interest in more esoteric mathematical structures (abstract algebras, pathological functions, transfinite numbers); with the study of non-Euclidean alternatives that detached mathematical geometry from an exclusive focus on physical space; and with the rise of kinetic theory, which converted the smooth differential equations of the 18th century into abstract approximations to physical phenomena actually discrete in the small. The bold new pure mathematics was free to pursue whatever seemed of mathematical interest, whether or not the results were of use in natural science. The application of mathematics then became a matter of selecting an effective model from among the wide range of abstract structures now being studied.[1]

While this new practice of pure mathematics produced an explosion of exciting new directions, the fundamental question -- what are we doing when we engage in mathematical thinking? -- was transformed into a far more difficult problem in the process. When mathematics is no longer a matter of investigating the underlying structure of the physical world, what is it? Two starkly different answers have occurred to reflective mathematicians. One of these, realism, echoes Plato in taking mathematics as the study of an independent realm of non-spatiotemporal, acausal abstracta.[2] Satisfying as this answer might be from the mathematician's perspective, philosophers have been troubled by questions about the nature and status of these abstracta and our epistemic access to them. The other response, if-thenism (or

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[1] This case is developed at greater length in [2008] or [2011], chapter 1.

[2] I have in mind here a version what's called 'Robust Realism' in [2011]. (As often happens, the canonical nod toward Plato here masks the complexity of his actual views.)

'deductivism'[3]), holds that contemporary pure mathematics is simply a matter of determining what conclusions follow from which assumptions. The various forms of realism have been widely discussed and debated in the philosophical literature, but if-thenism somewhat less so. My goal here is to explore its prospects -- and ultimately to propose a version that seems to me an attractive possibility.

I. Simple If-thenism

In the philosophical literature, if-thenism is often set in the context of the famous correspondence of 1899 to 1903 between Gottlob Frege and David Hilbert.[4] This fascinating exchange touched on many fundamental issues -- the nature of axioms and definitions, and the possibility of consistency proofs among them. Along the way, both isolated versions of if-thenism, 'although', as Michael Resnik notes, 'neither adopted it' (Resnik [1980], p. 106) -- Frege because he stuck to the realistic notion that axioms must be true, Hilbert because he felt the pull of finitism.

Around the same time, Bertrand Russell did adopt a version of the view --

> The typical proposition of mathematics is of the form '$\varphi(x, y, z, \ldots)$ implies $\psi(x, y, z, \ldots)$, whatever values $x, y, z, \ldots$ may have. (Russell [1903], p. 6)

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[3] Putnam [1967] calls it 'if-thenism'; Resnik [1980], chapter 3, Shapiro [2000], pp. 148-157, and Linnebo [2017], pp. 48-55, call it 'deductivism'. I follow Putnam's usage because it seems to me a more vivid way of capturing the core idea that the mathematician is only interested in what follows from what. (See footnote 8 for more on this terminology.)

[4] E.g., see Resnik [1980], chapter 3, and Shapiro [2000], pp. 148-157.

-- as did Hilary Putnam, with a nod to Russell, some decades later --

> Russell advocated a view of mathematics which he somewhat misleadingly expressed by the formula that mathematics consists of ‘if-then’ assertions. What he meant was not, of course, that all well[-]formed formulas in mathematics have a horseshoe as the main connective! but that mathematicians are in the business of showing that *if* there is any structure which satisfies such-and-such axioms (e.g. the axioms of group theory), *then* that structure satisfies such-and-such further statements (some theorems of group theory or other). (Putnam [1967], p. 20)

As it happens, neither Russell nor Putnam held to if-thenism for long: Russell soon turned to the logicism of *Principia Mathematica* and Putnam to a realism inspired by the indispensability arguments. Their reasons for defecting are addressed below, along with a host of other standard objections, but first a few clarifications of the position under consideration.

Because of the early association with thinking that led Hilbert to his formalism, and perhaps because logical implication itself is in some sense ‘formal’, there’s been a tendency to characterize if-thenism as involving the claim that mathematical statements are uninterpreted or ‘meaningless’:

> A *deductivist* … insists that … the practice of mathematics consists of determining logical consequences of otherwise uninterpreted axioms. The mathematician is free to regard the axioms (and the theorems) of mathematics as meaningless. (Shapiro [2000], p. 149)

This further component seems to me both inessential and ill-advised: there’s no obstacle to speaking of logical relations between meaningful statements, and to deny meaning to ‘2+2=4’ or the Fundamental Theorem of Calculus is to employ a notion of meaning at considerable remove from the way the term is ordinarily used in everyday life or linguistics.[5] Notice also that the stipulation of a universally quantified hypothetical is unnecessarily restrictive: it's natural to see a simple theorem of group theory as saying anything that's a group satisfies the conclusion, but the general goal of determining what follows from what is present in other contexts as well, for example, in axiomatic set theory, where we want to claim simply that the axioms logically imply the theorem.[6] Finally, some care must be taken in isolating the relevant axioms: only the 'axioms' of group theory (that is, the definition of 'group') are at play in a theorem of elementary group theory like 'every group has a unique identity', but some set-theoretic background is implicit, for example, in the classification of finite simple groups. In the first case, the if-thenist takes herself to have established one of those universally quantified theorems of logic -- if *x* is a group, then *x* has a unique identity -- in the second case, the logical implication is more complex -- if ZFC (or perhaps something less) and if *x* is a

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[5] Putnam appears to agree: the claim that the statements of infinitary mathematics 'are *meaningless combinations of signs* … does extreme violence to our intuitions. The view taken here … seems far closer to common sense' (Putnam [1967], p. 35).

[6] The literature on if-thenism is curiously wedded to the more restrictive formulation, insisting on versions like the above-quoted '*if* there is any structure that satisfies such-and-such axioms … *then* that structure satisfies such-and-such further statements' (Putnam [1967], p. 20) rather than the simpler 'such-and-such axioms imply such-and-such theorems' (ibid., p. 30). The latter seems to me a more natural rendition of the basic intuition that we're just trying to figure out 'what follows from what'. At one point, Putnam does raise the question of 'what it means to suppose that' structures exist, but he leaves this hanging and eventually opts for the idea that they are at least possible (ibid., p. 33). This way of thinking soon led to his modalism (Putnam [1967a], see also his [1979], p. xiii), latter elaborated in Helman's modal structuralism [1989]. 'Structuralism' does seem a more apt label.

finite simple group, then *x* is cyclic, alternating, Lie or sporadic'.[7,8]

As Resnik notes,

> Deductivism is a powerful and appealing philosophy of mathematics [that] avoid[s] commitments to abstract entities, and … simplifies … epistemology. (Resnik [1980], p. 136)

Of course it simplifies the epistemology of mathematics only by reducing it to the epistemology of logic, but there's no point addressing that challenge before we consider how if-thenism fares against some of fairly immediate objections. All philosophies of mathematics face serious, at least apparent, obstacles, and if-thenism is certainly no exception!

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[7] I think this is more-or-less Putnam's second suggestion (Putnam [1967], p. 22), minus the reference to structures (see previous footnote). He also confronts the objection that we only mean to be considering standard models of our 'ifs', which again strikes me as contrary to the spirit of if-thenism: e.g., ordinary first-order ZFC proves that all models of second-order ZFC of the same height are isomorphic and standard; some observers then worry that this proof could be taking place in a non-standard model of first-order ZFC, so those isomorphic models of second-order ZFC are really just non-standard in precisely the same way. But this purported problem only arises if we imagine there's some ambient, pre-theoretic semantics for the first-order ZFC in which we're carrying out our proofs, which no proper if-thenist would do. Something like this may be what Putnam means when he says that we only need a 'relational notion of non-standardness' (ibid., p. 24).

[8] One last note on terminology: though I use the term 'if-thenism' and speak of 'the antecedent of the if-then' and the like, I don't mean to assume that the theories whose consequences are being explored are finitely axiomatizable (as should be clear from the reference to ZFC). Those so-inclined might regard such cases as employing a purely disquotational notion of truth ('if the axioms of ZFC are true, then so-and-so is also true'), but the simpler route is just to regard the use of 'if-then' terminology here as a case of what mathematicians like to call 'abuse of notation'. I trust the meaning is clear.

II. Some common objections to if-thenism

Let’s begin with three traditional objections to if-thenism, only the third of which strikes me as serious enough to demand some substantial rethinking of the view.[9] The first of these objections is quite simple: if mathematics is just a matter of figuring out what conclusions follow logically from which assumptions, what was going on before the advent of modern axiomatic methods? If Newton and Euler, Fermat and Gauss isolated no explicit postulational or definitional setting, if their proofs weren’t always up to our standards of logical rigor, are we forced to say that they weren’t, after all, mathematicians?! Surely this would be an unacceptable consequence, a sad and sudden end for if-thenism!

Fortunately for the if-thenist, this objection can be turned away with a dash of entirely appropriate humility. Though philosophies of mathematics often claim, it seems to me unwisely, to account for all of past, present and future mathematics, if-thenism as characterized here only aims to describe the mathematics of the 20th and early 21st centuries, the practice as it arose out of the turn to pure mathematics over the course of the 19th. In earlier periods, especially the great flowering of analysis in the 18th century, the notion of ‘pure’ mathematics hadn’t yet arisen; as noted earlier, mathematics was seen as the investigation of the underlying mathematical structure of the physical world. The calculus of Newton and Leibniz was riddled with foundational mysteries, and its powerful

[9] More objections are treated along the way in later sections.

extensions in the hands of Euler and many others only exacerbated these problems, but the grounding of the resulting mathematics in empirical application pushed aside these concerns. As Lacroix famously remarked, 'such subtleties as the Greeks worried about we no longer need'.[10] The historian Morris Kline describes the situation this way:

> The physical meaning of the mathematics guided the mathematical steps and often supplied partial arguments to fill in nonmathematical steps. The reasoning was in essence no different from a proof of a theorem of geometry, wherein some facts entirely obvious in the figure are used even though no axiom or theorem supports them. (Kline [1972], p. 617)

But when pure mathematics was freed of any necessary link to the world, this guidance was gone and the character of the subject changed. If-thenism (as I understand it here) only aspires to describe this new way of doing mathematics.

While this response to the first objection is to remind ourselves of how mathematics was peeled off the world over the course of the 19th century, the second objection concerns what happens when parts of the new pure mathematics are subsequently applied in physical science. Intuitively, it seems that to apply '2+2=4' or the Mean Value Theorem, we need '2+2=4' itself, not 'the Peano Axioms imply that 2+2=4', we need the Mean Value Theorem itself, not 'ZFC implies the Mean Value Theorem'. To do their job in the service of true descriptions, don't the consequents themselves have to be unconditionally true?

When Putnam considers this question during his if-thenist phase, he focuses first on the use of simple arithmetical identities like our

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[10] Quoted in Kline [1972], p. 618.

‘2+2=4’ and concludes that it’s essentially a valid inference of first-order logic in disguise.[11] My if-thenist agrees.[12] Though Russell was apparently deterred from his if-thenism by simple mixed statements like ‘there are nine planets’, Putnam is happy to read these as purely logical claims as well.[13] But this approach doesn’t go far. Putnam’s broader if-thenist solution to the problem of how mathematics is applied involves the claim

> that mathematics is only used to derive statements in nominalistic [non-mathematical] language from statements in nominalistic language. (Putnam [1979], p. xiii)

Soon, Putnam’s embrace of indispensability considerations, derived from W. V. O. Quine, led him to reject this assumption, and if-thenism along with it.

The indispensability phenomena are well-known: mathematics plays an ineliminable role in our best-confirmed scientific theories; indeed, in many cases, these theories can only be stated in mathematical terms. While these observations are entirely accurate, they don’t seem to me to have the consequences for mathematical truth and ontology that have been claimed for them: confirmation from

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[11] Putnam [1967], pp. 27-29.

[12] She includes here identities involving sums, products and exponents (see below, around footnote 22). Once again the nature of logical truth is postponed to §IV.

[13] If ‘there are nine planets’ is understood to mean ‘if M is a model of *Principia Mathematica*, then the set of planets in M is a member of the 9 (the set of nine-element sets, defined in purely logical terms) of M’, as Russell seemed to think it must, then ‘we seem to be in trouble’ (Putnam [1967], p. 30): if there is no model of *Principia*, the number of planets could be any number at all! Of course this isn’t the only way to understand it -- there’s the familiar purely logical formulation -- as Putnam goes on to point out.

experience doesn’t rise uniformly through all aspects of a scientific theory, and much of the mathematization of science occurs in contexts that are explicitly or implicitly idealized or approximate.[14] In recent years, the argument has been recast in terms of explanation instead of confirmation, but it’s not clear to me that this changes the situation materially.[15]

The root of this difference of opinion on the viability of indispensability arguments lies, it seems to me, in two fundamentally contrasting visions of how mathematics works in scientific applications. To illustrate, consider Putnam’s dramatic appeal to the law of gravitation:

> To sketch the [indispensability] argument in a nutshell: … one wants to say that the Law of Universal Gravitation makes an objective statement about bodies -- not just about sense data or meter readings. What is the statement? It is just that bodies behave in such a way that the quotient of two numbers *associated* with the bodies is equal to a third number *associated* with the bodies. But how can such a statement have any objective content at all if numbers and ‘associations’ (i.e., functions) are alike mere fictions? It is like trying to maintain that God does not exist and angels do not exist while maintaining at the very same time that it is an objective fact that God has put an angel in charge of each star and the angels in charge of each of a pair of binary stars were always created at the same time! If talk of numbers and ‘associations’ between masses, etc. and numbers is ‘theology’ (in the pejorative sense), then the Law of Universal Gravitation is likewise theology. (Putnam [1975], pp. 74-75)

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[14] This case against the indispensability arguments is presented in [1997], §§II.6 and II.7, and revisited in [2007], pp. 94-96, 314-317.

[15] See Marcus [2014] for a recent entry in this debate, with a helpful overview of the literature. Note, in passing, that one of the central examples of a purported mathematical explanation is the famous life-cycle of the cicada: these come in prime numbers to ‘minimize the intersection of cicada life-cycles with those of both predators and other species of cicadas’ (Marcus [2014], p. 352). The ‘mathematics’ in this case is ’13 and 17 are prime’, which counts as a straightforward logical truth on the if-thenism being explored here (see footnotes 12, 22), and thus isn’t subject to if-then treatment in the first place. This response would seem to be open to Putnam as well.

The rhetoric here is undeniably persuasive, but let's pause a moment to ponder the case we've been offered.

Consider two bodies, say the earth and the sun. The Law of Universal Gravitation says:

$$F = G\frac{m_1 m_2}{r^2},$$

where m_1 is a number associated with the earth (its mass), m_2 is a number associated with the sun (its mass), r is the distance between them, G is a constant and F is the gravitational force between them. But now consider: what is r exactly? what is 'the distance between the earth and the sun'? It won't do to say we measure between the front surface of each, because the surface of the earth is quite bumpy in the large and ill-defined in the small, and the surface of the sun is even worse. In fact, as we learn in elementary physics, a nice theorem going back to Newton shows that the gravitational force exerted by spherical body acts as if all the mass were concentrated at its center, so in applying the Law of Universal Gravitation to the earth and the sun, we actually associate m_1 and m_2 with the points at their centers and take r to be the distance between those two points. But now we notice that 'the center of the earth' and 'the center of the sun' are also indefinite, and in fact, the proof of the nice theorem involves regarding the 'body' in question as a perfect sphere, breaking that sphere up into to a collection of infinitesimally narrow 'shells' (hence the name, 'the shell theorem'), then summing over these by integration.[16] As this point, the earth and the sun have

[16] Newton no doubt understood this process differently than we do today. There was considerable confusion, e.g., about 'physical infinitesimals' and

dropped cleanly out of the picture, replaced by mathematical abstracta. What this 'application' of the Law of Universal Gravitation actually describes is an abstract mathematical situation involving two perfect spheres equipped with continuous 'mass-density' functions and so on.[17]

This suggests a picture of applications quite different from Putnam's. He views the Law of Universal Gravitation, as applied to the earth and the sun, as a single claim that involves physical objects and properties -- the earth, the sun, their masses -- mathematical abstracta -- real numbers -- and functions between the two -- from masses to numbers, from spatial separations to numbers. The question then arises of how this instance of the law, as a whole, could be true if the mathematical abstracta it apparently refers to don't exist. In contrast, I've taken it to involve a completely mathematical model -- the earth and sun replaced by perfect spheres, masses by functions from those mathematical objects to real numbers, etc. -- and the applied mathematician's claim to be that this abstract model resembles[18] the worldly situation well enough to be used for

related notions, until the physical and the mathematical sorted themselves out with the rise of pure mathematics.

[17] Application of the Law of Gravitation to more complicated objects wouldn't involve perfect spheres, of course, and the calculus used would be more subtle than the shell theorem. But the point remains: the complicated physical objects would be replaced by pure geometric figures with precise boundaries, and the mathematics applied to these abstract surrogates.

[18] 'Resembles' here is intended only as a very broad catch-all for the sorts of relations applied mathematicians cite to explain why the idealizations in their mathematical modeling are benign and beneficial. E.g., water can be regarded as a continuous fluid because there's an intermediate point between a volume too small to have a stable temperature and a volume too large to have a uniform temperature; perfect billiard ball models of ideal gases work as well as they do because molecules have fairly stable effective radii; the

purposes of describing the gravitational force, that the simplifications and idealizations involved are beneficial and benign.[19] When we apply mathematics to the world, we sometimes know in detail how the abstract model relates to the physical situation (as in fluid dynamics), sometimes we know this only roughly (as in general relativity), and sometimes we have no clear idea at all (as in quantum mechanics). What's actually claimed about the world varies as this knowledge varies, but the general form is that the model resembles the physical situation well enough to be used for certain purposes.[20]

On this view of the application of mathematics, nothing at all follows about the existence of abstracta: just as we might describe the familial dynamics surrounding a powerful but aging person by comparison with 'King Lear', we can describe a given worldly situation by comparison with an abstract model. Pure mathematics doesn't need to deliver truths or existing entities in order to play this role any more than Shakespeare does.[21] All it needs to do is describe a

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earth can be replaced by a perfect sphere for reasons like those on the following footnote. It's not clear to me that any general account of 'resemblance' could cover the variety of such explanations in particular cases. (See, [2008], pp. 29-35, or [2011], pp. 21-30, for a related discussion.)

[19] Defending this claim involves pointing out that, e.g., that the variations on the earth's surface are negligible compared to the distance to the sun, so simplifying its shape to a physical sphere and replacing that physical sphere with a mathematical sphere shouldn't distort the situation unduly. Of course the ultimate defense is its empirical success.

[20] This is the intended upshot of the evolution described so quickly in the introductory paragraph above and at greater length in [2008] and [2011], chapter 1. See also [2011], §IV.2, for discussion of a related concern raised by Liston.

[21] Given this straightforward observation, it might be surprising that Leng [2010] expends so much effort explaining how her fictionalism about mathematical objects is compatible with applications. The explanation is

generous range of interesting structures and leave it to the physicist to determine which best serves which descriptive purpose.

This response to the second objection reveals that even our Simple If-thenism isn’t entirely simple. Not all statements of mathematics are regarded as disguised if-thens: elementary arithmetic claims, like 2+2=4, are actually logical truths, and the same goes for identities of arithmetic involving addition, multiplication, exponentiation, and their Boolean combinations.[22] (Exactly what this means for the application of elementary arithmetic to the world has to wait for the treatment of logic in §IV.) Though a common preference for generality produces many philosophies of mathematics that treat all parts, all branches, as essentially the same, there are unavoidable differences, and Simple If-thenism recognizes at least one of these. Furthermore, this important caveat considerably weakens some of the intuitive resistance to if-thenism: if-thenist’s opponent is robbed of the rhetorically powerful --‘but surely 2+2=4 is unconditionally true!’ -- leaving the somewhat less potent protest that the existence of a successor for every number shouldn’t be

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that she’s thinking of applications as Putnam does, of the Law of Universal Gravitation, for example, as involving both real physical objects and fictional abstracta. To make sense of this, she appeals to Walton’s theory of fictions with props, which allows her to import the actual earth and the sun as props into a fiction about real numbers and functions. That this instance of the law is a legitimate move in the fiction-with-props is an indirect way of saying something literally true about the physical situation. My suggestion is that the earth and the sun are actually replaced by abstracta in a purely fictional story, which is then compared with the purely physical situation.

[22] See [2007], pp. 318-319.

conditioned on the Peano Axioms or the Mean Value Theory on some ambient set theory.

On this view of applications, where pure mathematics describes an array of abstract models that may or may not end up being applied, a new question arises: obviously not every such description succeeds in specifying a coherent mathematical structure; in the brave new age of unfettered pure mathematics, free from the constraints of physical reality, how do we determine which structures are coherent, legitimate? The answer that emerged in the early 20th century, the answer that is now standard orthodoxy, holds that set theory is the final adjudicator: the legitimate structures are those modeled in V (the universe of sets).[23] To maintain the new freedom of mathematical inquiry, set theory should be as generous as it can be, and this provides one of the fundamental methodological commitments of set-theoretic practice.[24] For over a century, V has served this purpose well, first as codified by Zermelo, now with extensions including large cardinals (LCs). Of course, there’s no guarantee that this success will continue indefinitely -- this standard would no doubt fall if mathematically interesting and/or physically useful structures were located outside of V[25] -- but for now, this is how mainstream pure mathematics is practiced.

[23] See [2011], pp. 30-32. For a more nuanced account of set theory’s foundational role, see [2017], [201?].

[24] This admonition to ‘maximize’ is a central themes of [1997].

[25] Some argue that this has already happened, e.g., that the category of all groups or the category of all categories ought to be legitimate but isn’t supported by set theory. This raises the question of what exactly is being asked of set theory for the business of ‘founding’ (see [2017], [201?]), but

This brings us to the third, and I think most serious, common objection to if-thenism: if pure mathematics is just a matter of determining what follows from what, any consistent theory should be as good as any other in the 'if' part.[26] But this isn't how mathematics is actually done: 'ZFC + LCs' plays the fundamental role just sketched; the Peano Axioms, the Group Axioms, the axioms for any number of particular branches of mathematics, are all preferred over a vast store of alternative possibilities. Resnik puts the concern this way:

> The deductivist position on the choice of axioms conflicts with actual mathematical practice. According to the deductivist, it would be perfectly legitimate for mathematicians to make up axiom sets through some random method and then proceed to investigate their logical properties. But mathematics does not proceed in this way. … For example, we would not develop a set theory with the negation of the pair-set axiom, although it is possible to set up a set theory in which the negation of the pair-set axiom is consistent with the other axioms. (Resnik [1980], p. 132)

This objection seems to me entirely cogent: mathematics isn't just a study of what follows from what; it also involves judicious selection of preferred concepts and assumptions to play the 'if' role in its if-thens. Resnik's example involves an axiom -- why should the set theorist adopt the Axiom of Pairing? -- but many of the if-thenist's conditionals hinge on preferred concepts or definitions -- if *x* is a

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leaving these subtleties aside, Ernst has shown that no workable account of such 'unlimited categories' is possible -- e.g., there can be no category of all graphs in any such foundational scheme (see Ernst [2015]) -- so this isn't a strike against set-theoretic foundations in particular. Let me also note that even if set theory were removed from its foundational role, it would presumably continue as a branch of mathematics in its own right (though I'd expect that its proper methods might well shift in that circumstance).

[26] The paraconsistent logician would admit even inconsistent theories!

group, then *x* has a unique identity' -- and the corresponding question -- why should we study groups (rather than some odd but consistent variant)? -- is just as important, and just as troublesome to the Simple If-thenist. Let's now consider how the view might be enhanced to take these facts this into account.

III. Enhanced If-thenism

Resnik believes that what differentiates the preferred assumptions from the merely consistent ones is that they are true:

> Once one can distinguish between consistent axiomatic systems, one can make room for unconditional truth in mathematics. … the mathematicians themselves … say: 'Sure, set theory with the negation of the pairing axiom is consistent, but it is not true'. (Resnik [1980], p. 133)

Appealing to truth wouldn't immediately solve the problem of concept-formation, but let's leave that aside. Resnik imagines the if-thenist responding to this challenge by locating the source of mathematical preferences elsewhere: as a matter of 'the aesthetics of mathematics rather than its truth' (ibid.).[27]

This is certainly a style of response one hears, especially informally. All too often, it's invoked as a way of dismissing the question of what supports these preferences by removing it from the foundations or philosophy of mathematics and relegating to some other inquiry. That other inquiry might be systematic aesthetics (see below), but the dismissal often goes further, belittling the

[27] Following Quine and Putnam, Resnik eventually rejects this if-thenist response by appeal to the (purported) inseparability of mathematics from natural science (a line of thought I hope to have debunked in §II and the references cited there).

preferences as ‘merely aesthetic’, understanding this to entail ‘purely subjective’, and shifting the relevant questions to psychology or sociology. This might do for the mathematician, whose main goal is to deflect distracting philosophical challenges, but a philosopher or methodologist out to understand the practice of mathematics can hardly be satisfied: the processes of concept-formation and axiom choice are among the most far-reaching and consequential in all of mathematics; to legislate them out of mathematics entirely is just an unpersuasive redrawing of boundaries.

Still, mathematicians do sometimes make more thoughtful use of aesthetic language to describe what they do. For example, von Neumann writes that the contemporary pure mathematician …

> … has a wide variety of fields to which he may turn, and he enjoys very considerable freedom in what he does with them. … I think that it is correct to say that his criteria of selection, and also those of success, are mainly aesthetical. … mathematical ideas originate in empirics, although the genealogy is often long and obscure. But, once they are so conceived, the subject begins to live a peculiar life of its own and is better compared to a creative one, governed by almost exclusively aesthetical motivations, than to anything else, and, in particular, to an empirical science. (von Neumann [1947], pp. 2062-2063)

Similar remarks can be found even among natural scientists. For example, consider this from Dirac:

> It is more important to have beauty in one’s equations than to have them fit experiment … It seems that if one is working from the point of view of getting beauty in one’s equations, and if one has really sound insight, one is on a sure line of progress. (Dirac [1963], p.241)[28]

[28] The context of this quotation tells of Schrödinger’s early worries about his famous equation, when it seemed to conflict with experiment. Eventually, when electron spin was discovered, that conflict was shown to be merely apparent. In the omitted portion of the above quotation, Dirac remarks that ‘if Schrödinger has been more confident of his work, he could have published it some months earlier, and he could have published a more accurate equation’

Recently, some observers have made a positive efforts to understand how considerations of beauty might function and with what level of rationality (see Montano [2014], Kennedy and Väänänen [2015]).

I haven’t followed this path myself, partly out of ignorance of aesthetic theory, but also partly for methodological reasons: it seems to me best to start from the mathematics itself, to identify and evaluate the factors behind our preferences directly, in concrete cases, before considering any theoretical superstructure. All too often in the history of the subject, direct philosophical study of mathematics has been sidetracked by analogies -- mathematics is like natural science, mathematics is like a game, mathematics is like fiction -- when it could well turn out that an accurate picture of mathematics would see it as like nothing but its unique self.[29] In any case, the mathematical virtues that emerge in this sort of ground-up investigation might turn out to be ‘aesthetic’ in some important sense, and a general aesthetic theory might ultimately illuminate how this works, but at least to begin with, it seems to be best to allow the mathematics to speak for itself.

(ibid.). Apparently Schrödinger shared Dirac’s view on the centrality of mathematical beauty: ‘Schrödinger and I both had a very strong appreciation of mathematical beauty, and this appreciation of mathematical beauty dominated all our work. It was a sort of act of faith with us that any equations which describe fundamental laws of Nature must have great mathematical beauty in them’ (as quoted in Olive [1998], p. 89). See also: ‘The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty’ (as quoted in Pais [1998], p. 36).

[29] See Smith [2014] for some congenial skepticism that an aesthetic sense is what’s at work here. His case study concerns concept-formation (the fractional derivative).

So, on what grounds do we prefer some axioms, some concepts to others?[30] We can all agree, for example, that the concept of 'group' is preferable to many consistent alternatives.[31] It's commonplace to express this by saying that 'group' is a 'fruitful' concept, but we can in fact do better than that: we can specify what goals abstract algebra was out to serve ('to unify various seemingly diverse and dissimilar mathematical domains' (Kline [1972], p. 1157)), and note that 'group' does this job significantly better than the alternatives. To take an example that actually played a role in the historical development, an early switch from requiring cancellation laws to requiring inverses was motivated by the desire to bring important, newly discovered infinite groups into play. So the rationality of preferring 'group' to its near neighbors is a simple case of means-ends reasoning: if you want a concept to do this mathematical job, go for 'group'.

Similarly, if you want your set theory to found classical mathematics[32] -- in the weak sense, identified earlier, of providing an

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30 I hope I can be forgiven for passing quite briefly over the important considerations in the next three paragraphs. They are treated at length in [1997] and [2011].

31 For a bit more on the development of the group concept, with references, see [2007], §IV.3.

32 Linnebo ([2017], pp. 52-55) -- whose account of if-thenism incidentally follows Putnam in emphasizing structures over logical implication -- lodges this objection to the view: if-thenism is 'legitimate' for 'structural' axioms, like the group axioms, but not for 'foundational' axioms, like ZFC. The argument seems to be that set theory can't guarantee that a given theory is 'in good standing' (i.e., consistent) unless the axioms of set theory have some special status. But why can't we simply derive the consistency of the theory from the axioms of ZFC in ordinary if-then style? (This isn't a guarantee, of course, but we know better than to ask for more than relative consistency.) The discussion in the text indicates that for some axioms to be 'structural' (really definitions) and some 'foundational' (playing the

all-inclusive arena in which all coherent structures are realized[33] -- then you want it to be as generous as possible, so as to avoid any unnecessary limits on the reach of pure mathematics. Faced with a choice between Gödel’s Axiom of Constructibility (V=L) and Large Cardinal Axioms,[34] it’s enough to note that V=L rules out various structures that Large Cardinals admit, but that Large Cardinals preserve all the structures of V=L, now as the theory of the inner model L. Assuming sufficient evidence for consistency, since inconsistency scotches all benefits, there’s every reason to prefer ZFC+LCs to ZFC+V=L.

This line of thought takes our preferences for the ‘if’ part of the if-thens to be based in concrete mathematical facts and goals (not mere psychology or sociology) and explains the propriety, the rationality, of those preferences without appeal to truth (as the mathematical realist would have it). What’s wrong with Simple If-thenism is that it leaves out this whole realm of important mathematical activity, so the enhancement I propose is to add it back in: mathematics is a matter of figuring out what follows from what, where the concepts and axioms in the ‘if’ part are chosen with an eye to facilitating important mathematical goals.[35]

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role set theory plays) is just a matter of the mathematical jobs we devise them to do (see also [2007], pp. 352-355).

[33] As opposed to, say, a foundation that purports to provide the fundamental concepts and methods for all of mathematics. Again, see [2017], [201?] for more.

[34] ZFC+V=L implies that there are no measurable cardinals, so we can’t have both.

[35] In [2011], p. 99, this is called ‘sophisticated if-thenism’, and it bears obvious kindship to the Arealism proposed there for the special case of set

Happily, this Enhanced If-thenism also has the resources to turn aside yet another of the familiar objections to Simple If-thenism. The objection in question appears most forcefully in Quine's discussion of 'truth by convention' (Quine [1949]). He's considering the Huntington axioms for Euclidean geometry, which involve as primitives only 'sphere' and 'includes'; the rest are defined in terms of these. Now suppose that φ is some statement of Euclidean geometry expressed in these terms. Quine entertains the position that φ …

> … insofar as it is conceived as a mathematical truth, is to be construed as an ellipsis for 'if [the conjunction of the Huntington axioms] then φ', [which is] a truth of logic. (Quine [1949], p. 82)

This is Simple If-thenism. Quine goes on to object:

> The body of all such hypothetical statements, describable as the 'theory of deduction of non-mathematical geometry', is of course a part of logic; but the same is true of any 'theory of deduction of sociology', 'theory of deduction of Greek mythology', etc., which we might construct in parallel fashion with the aid of any set of postulates suited to sociology or to Greek mythology. (Quine [1949], p. 83)

Quine concludes that

> The point of view toward geometry which is under consideration thus reduces merely to an exclusion of geometry from mathematics, a relegation of geometry to the status of sociology or Greek mythology; the labeling of the 'theory of deduction of non-mathematical geometry' as 'mathematical geometry' is a verbal *tour de force* which is equally applicable to the case of sociology or Greek mythology … we are not interested in renaming. (op. cit.)

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theory. Of course there may well be other ways of 'enhancing' if-thenism, that is, other ways of accounting for our preferences that don't involve truth. For the sake of concreteness, the focus here is on this particular type of enhancement: means-ends assessment in pursuit of mathematical advantage.

Of course, the idea that there is something properly called ‘non-mathematical geometry’ with the same status as sociology, say, isn’t really a defect -- we do now believe that physical geometry is empirical -- but to put both physical geometry and sociology on a par with Greek mythology is more than merely troublesome. More to the point, in an effort to avoid questions of truth and existence, the Simple If-thenist turns mathematics into logic, but then seems unable to block the same move for everything else.[36] Surely, we don’t want to remove all, or even large chunks of empirical science from the realm of truth and existence![37]

Now consider this problem from the point of view of our Enhanced If-thenist: she reduces mathematics not just to what follows from what, not just to logic, but to logic supplemented with a mathematically rational process of selecting concepts and assumptions for the ‘if’ part of the if-thens. The counterpart for sociology or biology or physics would be to say that all its claims are really logical ‘if-then’s, and there’s a rational, properly sociological, biological or physical inquiry into what’s appropriate in the ‘if’ clause. This boils down to the fairly mundane observation that a science can be regimented as a set of assumptions and their logical consequences,[38] which isn’t particularly surprising; most of the effort in fact goes into isolating and defending those ‘assumptions’ for the

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[36] Following Quine, we engage here in the fantasy that axiomatizing a natural science is a straightforward affair, which of course it isn’t.

[37] Well, some do, but not me.

[38] Recall footnote 36.

‘if’ part -- formulating and confirming the theory in question -- and the ‘logical consequences’ part ordinarily goes without saying. Given the central role of proof in contemporary mathematics, there’s some point in this case to highlighting that aspect -- proving theorems is, after all, a large portion of what mathematicians do -- but the point remains that the same general format seems to work quite broadly. If it follows that truth and existence are irrelevant to mathematics, doesn’t the same follow for all that rest?

The answer lies in the ‘rational, properly sociological, biological, or physical inquiry’ that grounds preferences for the respective sociological, biological, or physical ‘if’ clauses: this is just a convoluted way of describing our empirical investigation into the sociological, biological, or physical facts. In these cases, we devise methods to uncover those facts -- beginning with ordinary observation, extending to controlled experimentation, theory formation and testing, and so on -- and we constantly reflect back on our methods to assess their reliability and to improve them. Our standard for assessing those methods is how well they track the truth about the existing worldly situations in question, be they sociological, biological, subatomic or cosmological. What makes mathematics different from the rest isn’t that it can be formed into if-then statements -- pretty much anything can[39] -- but the standards of assessment for what goes into the ‘if’ clauses -- these standards are different. Our decisions about the mathematical ‘if’ clauses involve

[39] Again, see footnote 36.

the local and global mathematical goals of the practice -- as group theory aims to draw together disparate structures in illuminating ways, as set theory aims to found classical mathematics -- but they don't involve the existence of abstracta or the truth of claims about them. So when mathematical if-thenism is enhanced, this doesn't yield the same ontological or epistemic conclusions as it does for sociology or biology or physics. Mathematics is different from the natural sciences.[40]

Interestingly enough, the Quine of 'Truth by convention' seems to reach this same conclusion -- that mathematics is different -- by a different route. Though he's rejected Simple If-thenism out of hand, he seriously entertains a variant: regard the conjunction of the Huntington axioms as true by convention; this, combined with the logical conventions,[41] yields our theorem φ as a true by convention as well. Then comes the familiar objection: this can also be done for 'the so-called empirical sciences' (Quine [1949], p. 100). Indeed,

> in thus elevating the statement from putative to conventional truth, we still retain the right to falsify the statement tomorrow if those events should be observed which would have occasioned its repudiation while it was still putative: for conventions are commonly revised when new observations show the revision to be convenient. (Quine [1949], pp. 101-102)

A few years later, in the classic essays 'Two dogmas' (Quine [1951]) and 'Carnap and logical truth' (Quine [1954]), considerations like

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[40] And from Greek mythology: settling the 'if' part in that case is essentially forming the myth; whatever constraints guided that process, they weren't the search for mathematical advantage.

[41] Of course the punch line of Quine [1949], which comes later in the paper, is that logic can't be treated as conventional, for Lewis Carroll-like reasons.

these lead Quine to question the very distinction between conventional and ‘putative’, but here, back in 1949, he takes a different tack:

> There is the apparent contrast between logico-mathematical truths and others that the former are a priori, the latter a posteriori. … Viewed behavioralistically and without reference to a metaphysical system, this contrast retains reality as a contrast between more and less firmly accepted statements … There are statements which we choose to surrender last, if at all … and among these there are some which we will not surrender at all, so basic are they to our whole conceptual scheme. Among the latter are to be counted the so-called truths of logic and mathematics, regardless of what further we might have to say of their status in the course of a subsequent sophisticated philosophy. (Quine [1949], p. 102)

This difference serves as motivation for the move to a conventionalist variant of if-thenism for the case of mathematics:

> Now since these statements are destined to be maintained independently of our observations of the world, we may as well make use here of our technique of conventional truth assignment and thereby forestall awkward metaphysical questions as to our a priori insight into necessary truths. On the other hand this purpose would not motivate extension of the truth-assignment process into the realm of erstwhile contingent statements. On such grounds, then, logic and mathematics may be held to be conventional while other fields are not; it may be held that it is philosophically important to circumscribe the logical and mathematical primitives by conventions of truth assignment but that it is idle elaboration to carry the process further. (Quine [1949], pp. 102-103)

And what does Quine himself think of this idea?

> Such a characterization of logic and mathematics is perhaps neither empty nor uninteresting nor false. (op. cit.)

Was there ever a more delightfully hedged assessment?! Of course, the denouement is that conventionalism cannot be sustained for logic, but it appears that the reduction of mathematics to conventions plus logic remains a live option for Quine at this point.[42]

[42] To be honest, I don’t see why Quine couldn’t have given the same style of response on behalf of the Simple If-thenist (in the passages quoted a few pages back from p. 83 of Quine [1949]), leaving that view live as well.

Though this early Quine and the Enhanced If-thenist may agree that this type of objection can be turned away with some version of the idea that mathematics is different from science, I hope it’s clear that those versions and the respective routes to them are not the same. Quine is motivated by a philosophical puzzle: mathematics is a priori; how are we to account for it, metaphysically and epistemologically? His conventionalist version of Simple If-thenism presents itself as a way of ‘forestalling’ these difficult questions. In contrast, the Enhanced If-thenist is simply out to describe mathematics as she sees it practiced: proving theorems from assumptions is perhaps the most conspicuous activity involved, but looking more closely she sees the importance of the selection of those ‘if’s. It’s only her examination and assessment of the methods there -- her developed understanding of what’s at stake, what guides and constrains these practices -- that reveals how fundamentally they differ from theory-formation and testing in the empirical sciences. So she and Quine agree that mathematics is different from science, but for very different reasons. As a result, they turn away the objection that natural science, too, is open to ‘if-then’ treatment in very different ways. He thinks there’s simply no motivation to extend that treatment to the natural sciences. She thinks we do so extend it when we codify an empirical science into an explicit theory, which is often something quite worth doing, but for her, that process itself reveals what separates mathematics from the rest: it’s in the methods of ‘if’ isolation, which float free of ontology and truth in mathematics but not in empirical science.

Assuming, then, that our Enhanced If-thenist has plausibly launched her attempted reduction of mathematics to logic and the isolation of ‘if’s, there remains the question left open at the end of §I: what is logical truth and how do we know it?

IV. The status of logic

Accounting for logical truth is a difficult problem for anyone, but Resnik suggests that it’s especially difficult for the if-thenist:

> Part of the usual justification for accepting a rule of deductive inference is that following it invariably leads from premises that are true … to conclusions which also are true. But since deductivism provides for no concept of mathematical truth beyond logical truth, it can be sure only of the correctness of the logical rules in their nonmathematical applications. Because such applications presumably are restricted to finite physical structures, … doubts about the use of logic in application to abstract infinite structures -- even those only hypothetically existing -- remain unanswered. (Resnik [1980], p. 136)

This is clearly an existential threat to if-thenism of any variety: what justification is there for the logical transition in the ‘if-then’? Let’s begin by thinking about the logic appropriate in direct application to the physical world (Resnik’s ‘finite physical structures’).

So, to begin, what is the logic of the world?[43] Our Enhanced If-thenist[44] looks around herself and discerns what certainly appears to be a world made up largely of individual objects -- stones, trees, stars -- with various properties -- the cat is black, the house is more than 100 years old -- standing in various relations -- Mount Everest is taller than Mount Baldy, the book is closer to the table than to the floor -- and some situations that depend on others -- this flower is larger than that one because I gave a well-balanced plant food to the first plant and not the second. Scientific examination of these various common sense beliefs largely ratifies them.[45] The result is a confirmed belief that the world exhibits a wide range of instances of what I've called KF-structure:[46] a domain of individuals, with properties, standing in relations, where some of these situations amount to conjunctions, disjunctions or negations of others, some hold universally, and some depend on others. There's every reason to suppose that these properties and relations are sometimes indeterminate -- there are tadpoles and there are frogs, but between

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[43] The next three paragraphs attempt to condense the position set out in Part III of [2007]. [2014a] is a more leisurely summary (with some additions). [2014c] explores the position in compare-and-contrast with Wittgenstein's views, both early and late.

[44] From here on, I often leave the 'Enhanced' implicit, trusting to the possessive 'our' and the capital 'I'.

[45] For the existence of these individual objects, I have in mind here investigations that show, e. g., that the atomic structure inside the area we take to be occupied by the stone is indeed quite different what's found outside, that various forces conspire hold these structures together as they move, to make them resist penetration, etc. Of course facts like these don't keep some from denying the existence of ordinary objects (see [2014a], pp. 95-97 and footnote 9, for some discussion of this point).

[46] For Kant-Frege, from whom the notion derives.

these there's a vague transitional phase; there's a fuzzy boundary between sloughing hairs that are still part of the cat and those that aren't -- and that this indeterminacy rises into compound situations in the familiar three-valued way: for example, not-(…) obtains if (…) fails, fails if (…) obtains, and is indeterminate if (…) is.[47] Since dependencies presuppose some connection or other between two situations, they can't be treated compositionally.

This much logical structure validates a rudimentary logic that includes many classical inferences involving 'not', 'and', 'or', and the quantifiers -- for example, double negation elimination, the DeMorgan laws, the distributive laws, universal instantiation -- as well as modus ponens; but there are no simple logical truths (e.g., (…) or not-(…) is indeterminate when (…) is) and some familiar inferences, like modus tollens, are invalid (e.g., if *p* grounds *q* and *q* fails, all that follows is that *p* can't hold -- it might be indeterminate). The power and simplicity of full classical logic is achieved by idealizing away both the indeterminacies, assuming all properties and relations are determinate, and ignoring the subtleties that separate dependencies from a material conditional. These idealizations are justified in a given application in just the way any scientific idealization is justified: if they provide significant benefits and don't distort the situation in ways relevant to the

[47] For the rest, (…) and (__) obtains if both do, fails if either does, and is otherwise indeterminate; (…) or (__) obtains if one does, fails if both do, and is otherwise indeterminate; (every *x*)(… *x* …) obtains if (… *a* …) does for every individual in the KF-structure, fails if there is an individual there for whom (… *a* …) fails, and is otherwise indeterminate. These clauses mirror the so-called Strong Kleene or Łukasiewicz connectives.

matter at hand. So it's not enough for a supporter of a deviant logic that rejects the law of excluded middle or installs a different style of conditional to point out that classical logic has falsified these things; that's what an idealization does. What's needed is a persuasive argument that the relevant falsification is damaging, perhaps in a particular kind of case, and that the proposed alternative logic is more effective than classical logic applied with sensitivity and care.[48] It isn't clear that such a case has ever been successfully made.

Given that the laws of rudimentary logic hold in any KF-structure, those laws are valid in any worldly situation with that structure. So, whenever the if-thenist has good evidence for the presence of KF-structuring, she has good evidence that the laws of rudimentary logic are straightforwardly truth-preserving -- and she has such evidence in many, many cases. (One conspicuous exception is the quantum world, where the apparent dependencies, properties and even objects don't behave in familiar KF-fashion, and there she finds rudimentary logic is unreliable.) In addition, given that the elementary arithmetical identities among the truths of logic (as in §II), we can now describe them more precisely as appearing among the

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[48] Williamson [1994] also advocates the use of classical logic despite the fact of vagueness, but he does so by embracing 'epistemicism': for borderline Joe, there's a fact of the matter about whether or not he's bald, we're just unable, in principle, to know which it is. My alternative suggestion is that we apply classical logic in vague situations with some care and sensitivity, just as we exercise care and sensitivity in applying any idealized theory: neglect friction carelessly and you predict that humans can't walk; neglect vagueness carelessly and you generate a sorites paradox.

literal validities of rudimentary logic, as literal truths about much of the world. This fills in the picture of applications from §II: much of applied mathematics enlists mathematical structures as abstract models that resemble phenomena in important ways, but elementary arithmetic straightforwardly describes the world's KF-structures. As for full classical logic, even in KF-structured cases, it needs to be applied with caution, like any idealized theory.

So, returning to Resnik, his worry begins from the assumption that 'logic' takes us from truths to truths when applied to finite physical structures; if 'logic' is understood as rudimentary logic and 'finite physical structures' as KF-structures, what we've seen so far is that our If-thenist agrees. The question is what justifies the use of logic in the mathematical if-thens, in connection with 'abstract infinite structures -- even those only hypothetically existing' (Resnik [1980], p. 136). We've seen how the Enhanced If-thenist offers an account of what properly guides our preferences for the 'if' clauses, locating the grounds for those decisions in means/ends rationality, and thus favoring 'ifs' formulated in terms of concepts like 'group' and set theoretic axioms like Choice and Large Cardinals. In her pursuit of such mainstream contemporary mathematics, this If-thenist also prefers classical logic to its various alternatives. What justification does she have for this preference?

The answer, it seems to me, is that 'hypothetically existing … abstract infinite structures' presented in the favored 'if' clauses have been imagined as straightforwardly KF. In fact, they are more precise than most real world KF-structures, because they admit no vagueness and the material conditional is entirely appropriate. This means that, as hypothesized, they are subject not only to rudimentary logic, but to full classical logic. So the appropriateness of applying classical logic in our thinking about them rests on the very same considerations that supported the relevant ‘if’ clause in the first place: structures like this are effective ways of realizing our mathematical goals, of gaining mathematical advantage. This doesn’t in any way rule out that other mathematical goals and other important mathematics could be facilitated, for example, by using a constructivist logic; there are clearly areas of mathematics that benefit from that alternative choice. Just as she embraces a number of different important options for her ‘if’ clauses, our If-thenist sees the merits of using different logics for different purposes, but the underlying criterion of selection is always the same: means/ends reasoning in pursuit of mathematical advantage.[49]

V. From logic to arithmetic

We’ve seen how the Enhanced If-thenist locates elementary arithmetical identities among the literal validities of rudimentary logic, not among the mathematical claims taken to be consequents of disguised if-thens. But it’s well-known that this logical treatment of arithmetic can’t be extended to its quantified claims: that no

[49] Obviously, this line of response isn’t open to other types of Enhanced If-thenist -- nothing much can be said without knowing how their distinctive ‘enhancements’ would go (see footnote 35) -- but a first thought would be that the preference for a given logic would be based on the same considerations as the preference for certain ‘ifs’.

number is smaller than zero, that addition is commutative, that no number has more than one successor. Many of these can be thought of as straightforward generalizations from the observed facts, but eventually we want a more systematic treatment. Presumably the world's KF-structures are all finite and indeed bounded by some large finite size -- the number of particles in the observable universe is thought to be smaller than 10^{80} -- so the ordinary facts, individual and general, about the world's number properties are also bounded. The trouble is that any attempt at a comprehensive theory limited to just this literal part of arithmetic would be too cumbersome to contemplate.

What we do instead is abstract away from the bounded finiteness of actual KF-structures. We figure that 'in principle', there could be arbitrarily large KF-structures, arbitrarily large numbers;[50] we then concoct an abstract mathematical model of that structure, with the '…' in '1, 2, 3, … ', as codified in the informal Peano Axioms (PA). In this way, elementary arithmetic -- a portion of rudimentary logic, grounded in the world, literally true -- is embedded in mathematical arithmetic, a branch of abstract mathematics. Of course, for the If-thenist, there is no actual model, no existing omega sequence; there's just the description, given by the informal Peano

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[50] Developmental research strongly suggests that we first come to believe that 'in principle' we can generate arbitrarily large numerical expressions, a conviction grounded in the recursive nature of the language-learning device, and infer from this that 'in principle' there could be arbitrarily large KF-structures (see [2014b], [2018]).

Axioms, of what such a structure would be like, installed in the 'if' of the arithmetic 'if-thens'.

In this way, our If-thenist's informal PA delivers as 'thens' such rudimentary truths as '2+2=4' and 'addition is commutative'. Idealized as it is, it also delivers various welcome falsehoods, like 'every number has a successor'. There's no a priori guarantee that the 'thens' of these 'if-thens' won't also deliver unwelcome falsehoods, for example, that it won't come into conflict with the rudimentary facts of elementary arithmetic. As with other cases of applied mathematical modeling -- for example, when the facts about actual liquids are embedded in an abstract mathematical theory of continuous substance -- the viability of embedding literal numerical truths in the abstract mathematical theory of the potential infinite must meet the test of experience; the model must demonstrate both its safety and its effectiveness. Like fluid dynamics, mathematical arithmetic has passed this test brilliantly. This is what justifies the Enhanced If-thenist's fixing on informal PA for the 'if' parts of her abstract systematization of worldly numerical facts.[51]

[51] [2014b] highlights some differences between the considerations that support PA and those that support, e.g., ZFC and its extensions. Though [2014b] also leaves open the possibility of attributing truth to arithmetic beyond the elementary, this isn't obligatory, so I ignore it here to focus on a stricter form of if-thenism.

VI. The status of meta-mathematics

With this understanding of arithmetic, we can face one final line of objection to if-thenism. It begins with the simple observation that it would be pointless to pursue an inconsistent theory. This might create some difficulty for a Simple If-thenist, but the Enhanced If-thenist naturally prefers consistent 'if' clauses, and the 'pointlessness' worry itself gives her good reason for this preference. The trouble is that the consistency of a given 'if', viewed meta-mathematically, is apparently itself a mathematical claim. Again Resnik gives this worry clear voice:

> If [the deductivist] is to believe that … some mathematical theories are non-trivial, he must commit himself to their unconditional consistency and thereby to some unconditional mathematical truths. (Resnik [1980], p. 119)

Concerns of this sort are exacerbated as meta-mathematics proceeds, with objections based on Gödel's incompleteness theorems, on the independence the Continuum Hypothesis (CH) from ZFC, and so on: if the Enchanced If-thenist doesn't believe the relevant meta-mathematical theorems are unconditionally true, why should she give up on an arithmetical proof of the Gödel sentence of first-order arithmetic or on settling CH in ZFC?

To evaluate this objection, we need to assess more carefully what our If-thenist actually believes about consistency. The question arises once she's installed informal PA in those 'if' clauses -- generating a practice of logically deriving 'thens', in natural language, with varying degrees of rigor and completeness. The consistency concern is just this: will this practice, properly executed, produce proofs with conflicting results; will it lead to the dead-end of a contradiction? Part of her case for the safety and effectiveness of this abstract systematization, noted at the end of the previous section, is her conviction that the answer is no. Why does she believe this? Perhaps at least partly because she has confidence that she and her fellows share a clear intuitive picture of a potentially infinite sequence that underlies informal PA,[52] that this picture is KF and thus amenable to classical logic, and thus that no harm should come of tracing ever more complicated consequences. She might also draw on the community's long experience with informal Peano Arithmetic, on her own first-hand familiarity with this and related systems, on the apparent absence of familiar sources of inconsistency, and the like. So far, though, this is an ordinary, somewhat tentative belief about the features of a particular human practice, not Resnik's properly mathematical belief.

The mathematical consistency claim doesn't appear until she begins to see, with Hilbert, the possibility of investigating this informal consistency more systematically, by mathematizing the situation. This is another an piece of applied mathematics: there's a worldly phenomenon of interest -- this time the activity of human mathematicians deriving consequences from informal PA -- which we undertake to study by concocting a mathematical model (for our If-thenist, a description of a model to fit in the 'if' clauses of a body of 'if-thens'). As so often happens when we prepare to apply mathematics, the worldly phenomenon is first simplified and regimented

[52] [2014b], [2018] take this picture to be grounded in the language-learning device (see footnote 50).

-- the ordinary language of this proving activity is replaced by certain strings of symbols ('formulas'), proofs by certain strings of strings of symbols ('proofs'), and so on -- but these are still physical objects (as Hilbert emphasized). If fact, they're straightforward KF-structures, just the sort of thing whose number properties have been embedded in the abstract arithmetic governed by informal PA. This time, though, instead of attending to their number properties, we focus on their structure as strings of symbols and strings of strings of symbols and set out to develop an informal theory of regimented informal proofs.

Just as in the case of number properties, the finiteness of the physical world means that these actual proof-theoretic properties form an unwieldy domain, and once again our systematization abstracts away from the physical or psychological limits to the lengths and complexities of the strings that appear in the practice. This time, we figure that PA's not leading to a contradiction isn't a matter of our limited interest or life span or any other such contingency; we think that PA is consistent 'in principle'. So we countenance indefinitely long formulas and proofs, embedding the ordinary facts in an abstract model that includes the '…' of the potential infinite. And, just as before, our If-thenist relies on a codified description of this model, on some natural list of axioms for her informal Proof Theory (PT), and in this way generates an applied mathematical theory that serves as a somewhat idealized but still effective and appropriate abstract model for the human practice of proving theorems from informal PA. This PT runs parallel to and independent of PA:

just as PA speaks of numbers and their successors and includes mathematical induction, PT speaks of strings of symbols, strings of strings of symbols, and their concatenations, and includes its own inductions on formulas and theorems. In this model, informal PA, expressed in natural language, is replaced by a regimented formalization, PA*, and informal provability from PA by PA*$\vdash$, which asserts the existence of certain strings of strings. With informal PT in the 'if' slot, our If-thenist generates 'if-then's like: if PT, then PA*$\vdash\forall x\exists y(y=Sx)$.[53]

At this point, the Enhanced If-thenist's informal consistency claim -- that no ordinary proof from informal PA will lead to a contradiction -- is transformed into the metamathematical claim: if PT, then PA*$\nvdash$1=0. The claim in the 'then' clause of this 'if-then' *is* a mathematical claim, a claim in the mathematically idealized, but still informal PT, perhaps the very claim that Resnik accuses the if-thenist of believing unconditionally. But does she? She's fairly confident of the informal consistency of informal PA, a non-mathematical claim, and her experience with informal PT leads her to believe that it's a reliable piece of applied mathematics, a mathematical model whose idealizations are both effective and benign. So the question is: does our If-thenist's confidence in the informal

[53] In a small abuse of notation, I implicitly assume that in contexts like 'PA*$\vdash$', the item on the right-hand-side of the turnstile has also been converted from ordinary mathematical language into a formal string of symbols, parallel to the move from PA to PA*.

consistency of PA commit her to similar confidence in the unconditional truth of the mathematical consistency claim?

The answer appears to be no. We've seen (in §II) that a mathematical model can do its job without any requirement of truth or existence, so simply modeling the informal consistency claim via the formal consistency claim isn't enough by itself to show that she must believe the latter. In fact, what our If-thenist actually believes is this: if PT is a good enough model to reflect this particular part of the actual practice of informally proving from PA, then the mathematical consistency statement will appear as the 'then' in an 'if-then' with PT as the 'if'.[54] But as for PA*$\nvdash$1=0 itself, it's still only appearing as the 'then' of an if-then.

So far so good for the Enhanced If-thenist, but Gödel's second incompleteness theorem appears to threaten from the wings.[55] We normally take it to show that we shouldn't bother trying to prove PA*$\nvdash$1=0 in PT -- more or less what Hilbert had hoped for, a proof-theoretic proof of consistency -- but if we don't unconditionally believe the theorem, why should we draw this moral?

In Gödel's hands, meta-mathematics matured beyond a simple study of (strings of) strings of symbols to the fully arithmetized

[54] This was more or less Hilbert's hope, to show that PT could prove PA*$\nvdash$1=0. His idea was that if we could prove this formal counterpart to informal consistency in our formal theory of proofs, this would increase our tentative confidence in informal consistency.

[55] There are anti-if-thenist arguments based on the first incompleteness theorem, but I don't think they raise any additional issues. (Some of these assume that the if-thenist is out to define, e.g., 'σ is true in arithmetic' as 'PA implies σ', but our If-thenist isn't defining 'mathematical truth', she's eschewing it.)

undertaking we know today. Combining informal PT with informal PA,[56] we define an encoding of strings as numbers; this allows us to generate strings like Fla(x), for codes of strings that are formulas of PA*, and Pbl(x), for codes of formulas for which there are proofs in PA*. We now have three levels of consistency claims: the worldly ‘there is no informal proof of 1=0 from informal PA’; the formalized claim in PT that $PA^* \nvdash 1=0$; and the equivalent arithmetized claim in PT+PA that $\neg Pbl(\#(1=0))$. The Enhanced If-thenist’s understanding of Gödel’s second theorem for PA* is then: If PT+PA, then $PA^* \nvdash 1=0$ implies $PA^* \nvdash \neg Pbl(\#(1=0))$.

What is our If-thenist to make of this? What morals should she draw from this theorem about the abstract model PT+PA? To begin with, PA and PT idealize real-world number properties and real-world strings in parallel ways, with the same characterization of the crucial ‘…’, so combining the two in a straightforward way shouldn’t change the Enhanced If-thenist’s assessment of the strengths and weaknesses of the idealizations involved. This means that, despite the fiction that numbers and formulas and proofs can be indefinitely large, she has in PT+PA what looks like an effective and benign model of actual number properties, statements, and proofs. Since it’s a good model, the fact that some claim holds in that model is fairly good indicator that the corresponding claim holds in the world. So the question is: for what

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[56] Notice that this combination, which I’ll refer to as ‘PT+PA’ below, isn’t just the joint list of their individual axioms: it also includes, e.g., the functions from the domain of PT to the domain of PA that are involved in Gödel numbering.

worldly situation is ‘PA*⊬1=0 implies PA*⊬¬Pbl(#(1=0))’ an abstract surrogate?

‘PA*⊬1=0’ is easy: this is the mathematical consistency claim, the model’s surrogate for the informal consistency of informal PA. PA*⊬¬Pbl(#(1=0)) takes just a little more unpacking: in PT+PA, ‘¬Pbl(#(1=0))’ is a coded equivalent of PA*⊬1=0,[57] so PA*⊬¬Pbl(#(1=0)) is the model’s arithmetized surrogate for ‘PT doesn’t informally prove PA*⊬1=0’.[58] Putting the two parts together, the worldly counterpart to ‘PA*⊬1=0 implies PA*⊬¬Pbl(#(1=0))’ is ‘if PA is informally consistent, then informal PT doesn’t informally prove PA*⊬1=0’. Insofar as we trust the model PT+PA, we should believe this worldly counterpart; since we also believe that PA is informally consistent, we shouldn’t bother trying to informally prove PA*⊬1=0 from PT. And this is exactly the moral we had hoped to draw.[59] The fact that PT doesn’t capture the formal counterpart (PA*⊬1=0) to something we believe unconditionally about informal proving from PA (that it won’t lead to 1=0) is a disappointment: we can trust what the model tells us, but we

[57] In PT+PA, it’s easy to see that Pbl(#(1=0)) iff there is an n in ω that codes a formal proof of 1=0 from PA* iff PA*⊢1=0.

[58] A complete version of this argument would require an explicit formulation of PT and PA+PT and a proof that PA*⊢¬Pbl(#(1=0)) iff PT*⊢(PA*⊬(1=0)). Thanks to Will Stafford for verifying that this can be done.

[59] Related considerations show that we shouldn’t expect to informally prove CH or not-CH from ZFC: insofar as we trust our formal theory of proving from ZFC, these worldly counterparts follow from the theorems of Gödel and Cohen.

can’t count on it to tell us everything. This is a shortcoming of the model, but not a challenge to the viability of Enhanced If-thenism.[60]

So in the end, none of the standard objections seems to me to be effective against Enhanced If-thenism. It recognizes the centrality of proof in contemporary pure mathematics as well as the subtle mathematical considerations that go into judicious choice of concepts and axioms. It supports the use of classical logic in most cases and preserves the unconditional truth of elementary arithmetic. What it doesn’t do is raise extraneous metaphysical or epistemological questions about abstracta or attempt to explain away serious mathematical decisions as arbitrary conventions. In short, it avoids both robust accounts of ‘mathematical truth’ and the threat of ‘anything goes’. For all this, I recommend it to your consideration.[61]

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[60] Again, if the if-thenist’s goal were to define ‘truth’ as ‘that which turns up in the relevant thens’, this would be a problem, but we’ve seen that our If-thenist has no interest in that project (see footnote 55).

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